MAGNETIC SUSCEPTIBILITY OF TWO DIMENSIONAL FREE ELECTRON GAS *

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ABSTRACT. The paramagnetic and diamagnetic susceptibilities of a two dimensional free electron gas have been calculated and the results compared with the susceptibilities of a three dimensional assembly.

The first application of Fermi-Dirac statistics was made by Pauli (1927) in discussing the paramagnetism of a free electron gas. It was shown by Landau[1] (1930) that on the basis of quantum mechanics there was also a diamagnetic contribution to the magnetic susceptibility, although on the basis of classical mechanics there should be no such effect. The paramagnetic effect arises due to the spin magnetic moment and the diamagnetic effect due to the quantization of the motion of the electron in a magnetic field. The temperature dependence of these effects has been investigated recently amongst others by Stoner[2] (1935). In the first part of the paper expressions for the paramagnetic susceptibility for the two dimensional case are obtained. The case of diamagnetism is discussed in section 2.

1. *Paramagnetism* :—In this section we shall consider the paramagnetism of a free electron gas and ignore the diamagnetism which is considered in the next section. When an external magnetic field H acts on the assembly of the electrons, the magnetic energy of an electron will have only two values—either $+H\mu$ if the spin is parallel to H, or $-H\mu$ if the spin is anti-parallel to H, where μ denotes the Bohr magneton and is given by

$$\mu = \frac{eh}{4\pi mc}. \qquad \ldots \quad (1)$$

The magnetic moment M produced by the magnetic field H can be determined as usual from F, the free energy, by the relation

$$M = -\left(\frac{\partial F}{\partial H}\right)_{T,V}, \qquad \ldots \quad (2)$$

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and we obtain, proceeding in the usual way, the result *

$$M_P = \frac{N\mu^2 H}{kT}\left[\frac{F'_{s-1}(\eta) + \frac{\beta_1^2}{6} F'''_{s-1}(\eta)\ldots}{F_{s-1}(\eta) + \frac{\beta_1^2}{2} F''_{s-1}(\eta)\ldots}\right] \qquad \ldots \quad (3)$$

where

$$\beta_1 = \frac{\mu H}{kT}, \quad F_s(\eta) = \int_0^\infty \frac{u^s}{e^{u-\eta}+1}\, du. \qquad \ldots \quad (4)$$

In the above expressions s has to be put equal to 3 for the three dimensional non-relativistic case, and equal to unity for the two dimensional non-relativistic case.

However, the magnetic moment can also be determined in a more straight forward way by using the Gibbs free energy instead of the Helmholtz free energy F, and this is equivalent to using the generalised form of Boltzmann's law. In the presence of the magnetic field, let N_+ denote the number of electrons in the assembly possessing spins parallel to H (magnetic moment opposite to H) and N_- possessing spins antiparallel to H, and let the difference between them be denoted by x, *i.e.*, $N_+ = N_- - x$. Then from a well-known theorem in thermodynamics† we have

$$\eta_+ - \eta_- = -\frac{2\mu H}{kT},$$

or

$$\log A_+ - \log A_- = -\frac{2\mu H}{kT}, \quad i.e., \quad \frac{A_+}{A_-} = e^{\frac{-2\mu H}{kT}} \qquad \ldots \quad (5)$$

where A_+ and A_- are the distribution parameters (absolute activities) for the N_+ electrons and N_- electrons respectively, and are related to η of the equation above by the relation $\eta_+ = \log A_+$, $\eta_- = \log A_-$.

We first consider the non-degenerate case.

In non-degeneracy $\frac{2\mu H}{kT} << 1$, and the terms in equation (5) can be expanded in a series. We have, since $\beta = +1$, in Fermi-Dirac statistics, inverting the series in (12) A.‡

$$A_+ = A_{1+}[1 + a_1 A_{1+} + a_2 A_{1+}^2 + \ldots], \qquad \ldots \quad (6)$$

* Stoner (1935). Stoner's result is for the special case of $s = \frac{3}{2}$-(Non-relativistic 3 dimensional case).

† $\eta kT = kT \log A$ is called the chemical potential.

‡ (12) A refers to equation (12) of the paper on 'Thermodynamic functions for two dimensional quantum statistics.' 'A' attached to an equation number will have the same significance in the rest of the paper.

$$A_{-}=A_{1-}[1+a_1A_{1-}+a_2A_{1-}^2+\ldots], \qquad \ldots (7)$$

where
$$a_1=\frac{1}{2^s},\ a_2=\left(\frac{2}{4^s}-\frac{1}{3^s}\right),$$

$$A_{1+}=\frac{2N_+}{D(kT)^s\Gamma(s)} \qquad \ldots (8)^*$$

$$=\frac{N_+}{N}2A_1\,. \qquad \ldots (9)$$

Similarly
$$A_{1-}=\frac{N_-}{N}2A_1\,.$$

Therefore,
$$\frac{A_+}{A_-}=\frac{N_+}{N_-}\left[1+\left(\frac{N_+-N_-}{N}\right)2a_1A_1+\left(\frac{N_+^2-N_-^2}{N^2}\right)4a_2A_1^2\right.$$
$$\left.+\left(\frac{N_-^2-N_+N_-}{N^2}\right)4a_1^2A_1^2\ldots\ldots\right]. \qquad \ldots (10)$$

But,
$$N_+=\frac{N}{2}\left[1-\frac{x}{N}\right] \text{ and } N_-=\frac{N}{2}\left[1+\frac{x}{N}\right].$$

Hence
$$\frac{A_+}{A_-}=1-\frac{2x}{N}\left[1+a_1A_1+(2a_2-a_1^2)A_1^2\ldots\right]+\frac{2x^2}{N^2}-\frac{2x^3}{N^3}\ldots, \qquad \ldots (11)$$

$$=e^{\frac{-2\mu H}{kT}}=1-\frac{2\mu H}{kT}+\frac{2\mu^2H^2}{k^2T^2}-\frac{4}{3}\frac{\mu^3H^3}{k^3T^4}\ldots\ldots, \qquad \ldots (12)$$

which gives, after a little algebra,

$$M=\mu x=\frac{\mu^2HN}{kT}\left[1-a_1A_1-2(a_2-a_1^2)A_1^2-\frac{1}{3}\frac{\mu^2H^2}{k^2T^2}\ldots\ldots\right], \qquad \ldots (13)$$

since x is the number of Bohr magnetons contributing to the magnetism of the assembly. In non-relativistic non-degeneracy, therefore, substituting the values of a_1 and a_2 for $s=\frac{3}{2}$, corresponding to the three dimensional case,

$$M_3=\frac{\mu^2HN}{kT}\left[1-\frac{A_1}{2^{3/2}}+\left(\frac{2}{3^{3/2}}-\frac{1}{4}\right)A_1^2-\frac{\beta_1^2}{3}\ldots\ldots\right], \qquad \ldots (14)$$

where
$$\beta_1=\frac{\mu H}{kT}.$$

* In three dimensions $\quad s=3/2 \quad$ and $\quad D=2\pi g\frac{(2m)^{3/2}}{h^3}V,$

and in two dimensions $\quad s=1 \quad$ and $\quad D=2\pi g\frac{m\sigma}{h^2},$

where σ represents the area, $\quad g=2$ for electrons.

The two asymptotic expressions (24) and (25) meet for $T=T_{02}$ and the expressions (26) and (27) meet for $T=T_{03}$.

It will be at once noticed that, for $T>T_{0\nu}$, the non-degenerate equation (24) and (26) will give a better approximation than (25) and (27). For $T<T_{0\nu}$, it is *vice versa*. $T_{0\nu}$ is called the degeneracy temperature. When $\nu=2$, *i.e.*, for the two dimensional case, we have from equation (22)

$$T_{02}=\frac{\epsilon_{02}}{k}=\frac{N}{Dk}=\frac{n^*h^2}{4\pi mk}, \qquad \ldots \quad (29)$$

and, for the three dimensional case, T_{03} is given by

$$T_{03}=\frac{\epsilon_{03}}{ks}=\frac{h^2}{4mk}\frac{n^{\frac{2}{3}}}{\pi^{\frac{2}{3}}3^{\frac{1}{3}}}. \qquad \ldots \quad (30)$$

Hence
$$\frac{T_{03}}{T_{02}}=\frac{n^{\frac{2}{3}}}{n^*}\left(\frac{\pi}{3}\right)^{\frac{1}{3}}, \qquad \ldots \quad (31)$$

where T_{03} is the degeneracy temperature in three dimensions and T_{02} in two dimensions.

2. *Diamagnetism* :—We consider the case in which the electron gas is restricted to move in a plane, the magnetic field H being at right angles to this plane. The electrons move in circular orbits and the possible values of energy for any electron (excluding spin energy) are given by $\epsilon_l=2\mu H(l+\frac{1}{2})$, where l is any integer.

The number of states with the given quantum number l is determined in the following way. When the field is applied, the electrons group themselves into the nearest quantized energy level. The condition that this energy level has the quantum number l is that the energy $\frac{p^2}{2m}$ of the electrons in the absence of the field (p is the momentum of the electron in the plane) lies in the range

$$2\mu Hl\leq\frac{p^2}{2m}\leq 2\mu H(l+1).$$

And hence, if σ denotes the area of the plane containing the electrons, the corresponding area of phase space is $2\pi\sigma p dp=4\pi\sigma m\mu H$. Allowing the weight factor g for each h^2 of phase space, we have the number of states possible

$$=g.\frac{4\pi\sigma m\mu H}{h^2}=g.\frac{eH\sigma}{ch}. \qquad \ldots \quad (32)$$

Therefore*, $$\Omega=\frac{-kTgeH\sigma}{ch}\sum_{l=0}^{l=\infty}\log\left\{1+e^{\eta-(l+\frac{1}{2})\frac{2\mu H}{kT}}\right\}.$$

* Stoner, *loc.cit.* Ω is the partition function.

Using Euler's formula $\sum_{l=a}^{l=b-1} f(l+\frac{1}{2}) = \int_a^b f(x)dx - \frac{1}{24}\left| f'(x) \right|_a^b + \ldots\ldots\ldots,$

we have, $\Omega = -kTg\frac{eH\sigma}{ch}\left[\int_0^\infty \frac{2\beta_1 x}{e^{2\beta_1 x-\eta}+1}dx + \frac{\beta_1}{12}\left(\frac{1}{e^{-\eta}+1}\right)\ldots\right],$ ... (33)

where $\beta_1 = \frac{\mu H}{kT}$. Putting $2\beta_1 x = u$ and simplifying,

we have $$\Omega = -\frac{2\pi mg\sigma}{h^2}(kT)^2\left[F_1(\eta) - \frac{\beta_1^2}{6}\frac{1}{e^{-\eta}+1}\ldots\right]. \quad \ldots (34)$$

Therefore, $$N = -\frac{1}{kT}\frac{\partial\Omega}{\partial\eta} = \frac{2\pi mg\sigma}{h^2}kTF_0(\eta), \quad \ldots (35)$$

and $$M_D = -\frac{\partial\Omega}{\partial H} = \frac{2\pi mg\sigma}{3h^2}\mu^2 H\frac{1}{e^{-\eta}+1}$$

$$= -\frac{N}{3}\frac{\mu^2 H}{kT}\frac{1}{F_0(\eta)}\frac{1}{e^{-\eta}+1}. \quad \ldots (36)$$

In non-degeneracy $A = e^\eta << 1$, $F_0(\eta) = A$, and therefore to a first order

$$M_D = -\frac{1}{3}\frac{N\mu^2 H}{kT}. \quad \ldots (37)$$

In degeneracy $\eta >> 1$, $F_0(\eta) = \eta$ and to a first order

$$M'_D = -\frac{1}{3}\frac{N\mu^2 H}{kT\eta}. \quad \ldots (38)$$

But from equation (22), for $\eta >> 1$,

$$\epsilon_{02} = \eta kT = \frac{Nh^2}{g.2\pi m\sigma},$$

and therefore, in degeneracy,

$$M'_D = -\frac{1}{3}\frac{g.2\pi m\sigma}{h^2}\mu^2 H. \quad \ldots (39)$$

Hence the particle susceptibility for two dimensions is given by $\chi_2 = -\frac{1}{3}\frac{\mu^2}{kT}$ for the non-degenerate case and $\chi'_2 = -\frac{\mu^2}{3}.\frac{4\pi m}{h^2 n^*}$ for the degenerate case, where n^* is the number of electrons per unit area.